The Disjoint Paths Problem

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1 Introduction

A basic problem that arises in large-scale communication networks is that of assigning paths to *connection requests*. Each connection request is a pair of physically separated nodes that wish to communicate over a path through the network; given a list of such requests, one wants to assign paths to as many as possible in such a way that no two paths "interfere" with each other. Thus, for a given list of requests, we can ask a number of natural questions. How many requests are simultaneously *realizable* using paths that are mutually edge-disjoint? How many *rounds of communication* are required to satisfy all requests, when all paths assigned in a single round must be edge-disjoint? These turn out to be classical NP-complete problems; previously known approximation techniques for these problems are limited either to very special graphs, or to "high-bandwidth" models in which a large number of paths can share a single edge.

The intractability of the disjoint paths problem does not appear to be simply a theoretical phenomenon. Although the types of problems that arise in the context of *virtual circuit routing* tend to have additional side constraints (e.g. connections have limited duration and can bring varying amounts of "profit"), the formulation described in the first paragraph contains the essence of virtually all such routing problems in which each connection consumes a large fraction of the bandwidth on a link. As such, the current lack of understanding of the disjoint paths problem is a major obstacle to the design of practical heuristics. Indeed, [4] notes that in practice, the greedy algorithm tends to be used for routing, despite its bad performance on a number of very common interconnection patterns. Moreover, robust ways are known for converting algorithms for the "pure" formulation into algorithms that can handle connections of limited duration or variable value [3]; thus, the difficulties contained in these more elaborate routing problems seem to stem mainly from the complexity of the disjoint paths problem.

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2 The Maximum Disjoint Paths Problem

Since our problem is NP-complete, it is natural to look for efficient approximation algorithms. To be precise, let $G = (V, E)$ be a graph on n vertices and $\mathcal{T} = \{(s_1, t_1), \ldots, (s_k, t_k)\}$ a collection of *terminal pairs* — pairs of vertices of G. We say that $\mathcal{T}$ is *realizable* in G if there exist mutually edge-disjoint s_i-t_i paths, for $i = 1, \ldots, k$. We can now define three natural optimization problems, all of them NP-complete.

(i) The maximum disjoint paths problem (MDPP). We wish to find a maximum size realizable subset of $\mathcal{T}$.

(ii) The on-line MDPP. Terminal pairs arrive in an arbitrary order; when presented with each pair we must decide immediately whether to reject it or route it on a path through G.

(iii) The minimum rounds problem. We wish to find the minimum number of *rounds* of communication needed to route all of $\mathcal{T}$, where pairs routed in a single round must use edge-disjoint paths. This is equivalent to determining the minimum number of wavelengths required for routing $\mathcal{T}$ in the optical model discussed in [11, 1].

Much of the previous work on these problems has dealt with the case in which each path consumes only a small fraction of the available bandwidth on an edge; this can be modeled by requiring $\Omega(\log n)$ parallel copies of each edge. Here, the randomized rounding technique of Raghavan and Thompson [10, 9] can be used to obtain good approximations for the MDPP; on-line algorithms for this case are considered by Awerbuch et. al. [2].

As noted in [4] however, there are many applications in which each communication path consumes a large fraction of the available bandwidth on a link; thus it makes sense to consider approximation algorithms for graphs without a large number of parallel edges. The results here are much more restricted. For trees with parallel edges, Garg, Vazirani, and Yannakakis [6]

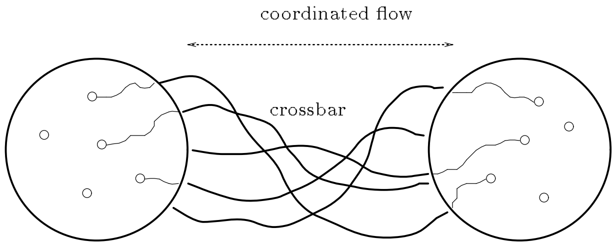

Figure 1: Routing between two clusters in a graph

obtain an off-line 2-approximation. Awerbuch et. al. [3, 4] give randomized on-line algorithms for trees. The MDPP in graphs with strong expansion properties is studied by Broder, Frieze, and Upfal [5], and others.

3 Our Results

Jointly with Éva Tardos [7, 8], we have been working on approximation algorithms for the MDPP in general classes of graphs. We define the class of *densely embedded graphs*, which includes the two-dimensional mesh and other common planar and locally planar interconnection networks. Roughly speaking, a *densely embedded graph* G is one which can be drawn "densely enough" on a surface so that within $\log n$ steps of each node, the graph is planar, and has the "mesh-like" property that any set of k nodes is joined by at least $\Omega(\sqrt{k})$ edges to the rest of G. We further require that aside from a few *exceptional faces*, the faces of G in the embedding have bounded size, and all vertex degrees are even — again, these are properties of the mesh. (We refer to the latter requirement as a *nearly-Eulerian* condition.) Our main results are:

- We obtain a constant-factor off-line approximation algorithm for the MDPP in densely embedded nearly-Eulerian graphs [8]. This improves on an $O(\log n)$-approximation for the special case of the mesh due independently to Aumann–Rabani [1] and our earlier paper [7]; for networks that are not explicitly required to be "high-capacity," it is also the first constant-factor approximation for the MDPP in any class of graphs besides trees.

- We obtain an $O(\log n)$-competitive randomized on-line algorithm in the same class of graphs [8]. This improves on an $O(\log n \log\log n)$-competitive algorithm of Awerbuch, Gawlick, Leighton, and Rabani for the special case of the mesh [4]; moreover, by a lower bound of [4], our algorithm is asymptotically optimal.

- Our constant-factor approximation for the MDPP automatically gives an $O(\log n)$-approximation for the minimum rounds problem; this improves on an $O(\log^2 n)$-approximation for the mesh [1]. For a related class of planar graphs, we can obtain a constant-factor approximation for the minimum number of rounds when all terminal pairs are roughly the same distance apart in G [7].

A basic subroutine in our earlier paper [7] is that of routing connections between two *clusters* in a graph; a cluster here is essentially a k-step neighborhood of a vertex, for some k. (See Figure 1.) To accomplish this, we must route corresponding ends of each connection to the cluster boundaries, and then link up these ends appropriately. In the mesh, the latter problem is solved conveniently by making use of *crossbar* structures — large configurations of paths that mutually cross. We show that crossbars in fact exist in any nearly-Eulerian planar graph, and can be used in much the same way.

The other problem is the following: we don't want to route s_1 to a cluster boundary only to find that, in the cluster containing its mate t_1, we have instead routed t_2 to the boundary. To handle this, we define the notion of a *coordinated flow*, which only routes s_i in one cluster if t_i is routed in the other. Coordinated flows turn out to have some useful matroidal properties; in particular, the minimum rounds problem can be solved in some cases via matroid covering algorithms.

In [8], we make use of some of these techniques as part of an algorithm that builds a "simulated network" on the clusters; within this simulated network we can apply variants of the algorithms of [10, 9, 2].

References

[1] Y. Aumann, Y. Rabani, "Improved bounds for all-optical routing," *Proc. 6th ACM-SIAM SODA*, 1995.

[2] B.Awerbuch, Y.Azar, S.Plotkin, "Throughput-Competitive Online Routing," *Proc. 34th IEEE FOCS*, 1993.

[3] B. Awerbuch, Y. Bartal, A. Fiat, A. Rosén, "Competitive non-preemptive call control," *Proc. 5th ACM-SIAM SODA*.

[4] B. Awerbuch, R. Gawlick, F.T. Leighton, Y. Rabani, "On-line admission control and circuit routing for high performance computing and communication," *Proc. 35th IEEE FOCS*, 1994.

[5] A. Broder, A. Frieze, E. Upfal, "Existence and construction of edge-disjoint paths on expander graphs," *SIAM J. Computing*, 23(1994), pp. 976–989.

[6] N. Garg, V. Vazirani, M. Yannakakis, "Primal-dual approximation algorithms for integral flow and multicut in trees, with applications to matching and set cover," *ICALP*, 1993.

[7] J. Kleinberg, É. Tardos, "Approximations for the disjoint paths problem in high-diameter planar networks," *Proc. 27th ACM STOC*, 1995.

[8] J. Kleinberg, É. Tardos, "Disjoint paths in densely embedded graphs," submitted.

[9] P. Raghavan, "Probabilistic construction of deterministic algorithms: approximating packing integer programs," *J. Computer and System Sciences*, 37(1988), pp. 130–143.

[10] P. Raghavan, C.D. Thompson, "Randomized rounding: a technique for provably good algorithms and algorithmic proofs," *Combinatorica*, 7(1987), pp. 365–374.

[11] P. Raghavan, E. Upfal, "Efficient all-optical routing," *Proc. 26th ACM STOC*, 1994.